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PERIODS OF THE SOLUTIONS OF A DIFFERENCE EQUATION IN $\mathbb{R}_*^+$ ASSOCIATED TO A SPECIAL QRT-MAP

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Abstract. We study the periods of the solutions of the difference equation in $\mathbb{R}_*^+$

$$u_{n+1}u_n = av_n + b, \qquad v_{n+1}v_n = u_{n+1} + \frac{b}{u_{n+1}}, \qquad a > 0,\ \ b > 0, \tag{1}$$

associated to the special QRT-map given by the family of elliptic cubic curves $\mathcal{C}_{a,b}(K)$ with equations

$$xy^2 + x^2 + ay + b - Kxy = 0. \tag{2}$$

Using Weierstrass' functions we calculate the rotation number of the restriction of the associated QRT-map to the positive part of a cubic $\mathcal{C}_{a,b}(K)$ of the family (2). Using the prime number theorem, we prove that every integer $n \geq 1$, excepted 2, 3, 4, 6, 10, is the minimal period of some solution of (1) for some $a > 0$ and $b > 0$. The exceptional numbers are minimal periods of no solution of (1), whatever positive values have the parameters a and b.

Keywords. Difference equations; elliptic curves; QRT maps; rotation number; periodic orbits.

AMS (MOS) subject classification: 39A20, 39A11

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