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THE CHEN SYSTEM REVISITED

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Abstract. This note points out that the assertions of ["Chen's attractor exists if Lorenz repulsor exists: The Chen system is a special case of the Lorenz system," CHAOS 23, 033108 (2013)] are groundless and incorrect. The failure of that criticism actually supports the strong standing of the Chen system.

Keywords. Chen system, Lorenz system, chaos, attractor.

The celebrated general parametric Lorenz system discussed in [1] and herein is described by

$$\begin{aligned} \frac{dX}{dt} &= \sigma(Y-X) \\ \frac{dY}{dt} &= \rho X - Y - XZ \\ \frac{dZ}{dt} &= -\beta Z + XY \end{aligned} \tag{1}$$

where σ, ρ and β are real parameters, while the classical (original) Lorenz system refers to

$$\begin{aligned} \frac{dX}{dt} &= 10(Y-X) \\ \frac{dY}{dt} &= 28X - Y - XZ \\ \frac{dZ}{dt} &= -\frac{8}{3}Z + XY \end{aligned} \tag{2}$$

which is chaotic on a small subset[1] $\{\sigma, \rho, \beta\} = \{10, 28, 8/3\}$ inside the 3D real parameter space of the general form (1), but for other parameter sets system (1) may not be chaotic.

[1]Due to the structural stability of the Lorenz system, this parameter set can be somewhat enlarged, but using this set does not affect the discussions throughout. The same remark applies to system (4).

The so-called general parametric Chen system discussed in [1,2] and also herein is described by

$$\begin{aligned} \frac{dx}{dt} &= a(y-x) \\ \frac{dy}{dt} &= (c-a)x + cy - xz \\ \frac{dz}{dt} &= -bz + xy \end{aligned} \tag{3}$$

where a, b and c are real parameters, while the chaotic case is with the parameter set $\{a, b, c\} = \{35, 3, 28\}$, giving

$$\begin{aligned} \frac{dx}{dt} &= 35(y-x) \\ \frac{dy}{dt} &= -7x + 28y - xz \\ \frac{dz}{dt} &= -3z + xy \end{aligned} \tag{4}$$

Likewise, for other parameter sets of $\{a, b, c\}$, system (3) may not be chaotic.

In [1], it is suggested that, with $c \neq 0$, the linear transform

$$x = -cX, \quad y = -cY, \quad z = -cZ, \quad \tau = -ct \tag{5}$$

which has reversed the time variable if $c > 0$, can convert the general Chen system (3) to

$$\begin{aligned} \frac{dX}{d\tau} &= \tilde{\sigma}(Y-X) \\ \frac{dY}{d\tau} &= \tilde{\rho}X - Y - XZ \\ \frac{dZ}{d\tau} &= -\tilde{\beta}Z + XY \end{aligned} \tag{6}$$

where

$$\tilde{\sigma} = -\frac{a}{c}, \quad \tilde{\rho} = \frac{a}{c} - 1, \quad \tilde{\beta} = -\frac{b}{c} \tag{7}$$

By carelessly looking at the algebraic forms of the equations, it is easy to believe that system (6) is exactly the same as system (1), or a special case of it as asserted in [1].

First of all, it must be pointed out that the algebraic form and the dynamical property of a system are two different concepts and issues: system $\dot{x} = ax$ with $a = 1$ and system $\dot{x} = \tilde{a}x$ with $\tilde{a} = -1$ have the same algebraic form, but the former is unstable while the latter is stable about their zero equilibrium. In dynamical systems theory, time reversion is a very critical operation which typically change the dynamics of a system (for example, the time reversal of the above two simple systems change the system stability to the opposite). In the present discussion, if the following simple form of transformation (5) is used on the chaotic Lorenz system (2):

$$x = X, \quad y = Y, \quad z = Z, \quad \tau = -t\,,$$

one obtains a divergent system with all equilibrium stabilities changed to the opposite, which is by no means equivalent to the original Lorenz system (2). This also explains why the concept of "equivalence" was never defined through time reversion in all the chaos theory literature; otherwise, one will be led to a wrong assertion that the two systems $\dot{x} = x$ and $\dot{x} = -x$ are "equivalent".

Second, it must also be pointed out that although the parameters set $\{\sigma, \rho, \beta\}$ of system (1) can be the entire 3D real space, for this nonlinear system its dynamics vary from region to region in the parameter space; therefore, just like other nonlinear systems, system (1) is typically being investigated case by case on different parameter sets. Observe, for example, if system (1) is defined only on a restricted parameter set with $\beta > 0$, which is the case of the chaotic Lorenz system (2) where $\beta = 8/3$, its third equation $\frac{dZ}{dt} = -\beta Z + XY$ is stable about zero on the two planes $X = 0$ and $Y = 0$, while if system (6) is defined only on a restricted parameter set with $\tilde{\beta} < 0$, which is the case of the chaotic Chen system (4) where $\tilde{\beta} = -3/28$, the corresponding third equation $\frac{dZ}{d\tau} = -\tilde{\beta} Z + XY$ is unstable about zero on both planes $X = 0$ and $Y = 0$. So, one can see the subtle and essential difference between the two systems when they are defined on different parameter sets.

Next, observe that although the title, abstract, introduction and conclusion of [1] all try to create an impression to the readers that the Chen system is equivalent to (or a special case of) the Lorenz system no matter what, in the technical text of [1] it is frankly made clear that the two systems (1) and (3) are "equivalent" (after time reversion) on and only on a particular parameter set, which is merely a plane in the 3D real parameter space (or a line in the 2D real parameter plane which [1] discusses):

$$\rho + \sigma = -1 \tag{8}$$

The original statement given in [1] is quoted here for clarity: "Therefore, generally, for $c \neq 0$, the Chen system is equivalent to the Lorenz system in the particular case of the parameter plane $\rho + \sigma = -1$."

However, obviously many different systems can be equivalent or even identical if they both are restricted (projected) on a lower-dimensional (parameter) subspace, hence the above "equivalence" does not explain anything meaningfully. Even so, an easy catch is that the chaotic Lorenz system (2) does not satisfy this condition at all, where $\rho + \sigma = 28 + 10 \gg -1$.

Nevertheless, it is inferred in [1] that the general Lorenz system (1), when its parameters are restricted to a subset satisfying condition (8) which does not include the Lorenz system (2), is "equivalent" to the chaotic Chen system (4). We do not follow this logic of inference, since a non-chaotic system may be similar or even identical to a chaotic system in their algebraic forms but they are never equivalent in terms of dynamics, once again due to the difference in their defining parameter sets.

In the study of chaos theory, one is particularly interested in the case where both systems are chaotic. It has been proved in [3] that the chaotic Lorenz system (2) is not smoothly-equivalent to the chaotic Chen system (4), namely there does not exist a smooth transform of variables that maps one system to another. This statement remains intact unless the proof given in [3] is found erroneous or one could show a precise counterexample.

To proceed further, in [1] a so-called "Lorenz repulsor" is constructed, as follows: remove the Lorenz chaotic parameter set $\{\sigma, \rho, \beta\} = \{10, 28, 8/3\}$ from system (1) or (2), and replace it by the Chen chaotic parameter set $\{a, b, c\} = \{35, 3, 28\}$, then apply transformation (7) to result in a "Lorenz repulsor" which is in the form of (6) with $\tilde{\sigma} = -a/c = -35/28$, $\tilde{\rho} = 35/28 - 1$ and $\tilde{\beta} = -b/c = -3/28$ (see the Caption of FIG. 2 in [1]). It is clear that this "Lorenz repulsor" is nothing but a linear transformation of the chaotic Chen system (4), which is not obtained from the chaotic Lorenz system (2) per se. In other words, the assertion of [1] is equivalent to saying that "if the attractor of a linearly-transformed Chen system (4) exists, then the attractor of the original Chen system (4) exists".

Thus, up to this point, according to [1], we are given the following statements and facts:

1. The chaotic attractor of the Lorenz system (2) (with parameters in a small neighborhood of $\{\sigma, \rho, \beta\} = \{10, 8/3, 28\}$) exists by the well-known result of Tucker [4];

2. The parametric Lorenz system (1) with parameters satisfying $\rho + \sigma = -1$, which does not include the chaotic Lorenz system (2), is "equivalent" to the chaotic Chen system (4);

3. The existence of the attractor of the "Lorenz repulsor", precisely a linear transformation of the Chen system (4) but not of the Lorenz system (2), implies the existence of the attractor of the Chen system (4).

Based on the above, it is asserted in (e.g., the Title of) [1]: "Chen's attractor exists if Lorenz repulsor exists." Obviously, no one is able to follow this logic of inference. The assertion is groundless and incorrect.

It should be noted that actually we were the first to point out that the Chen system is a special case of a generalized Lorenz canonical form (see, e.g., [5] and some other references cited in [1]). In some of our publications (again, see the references cited in [1]), we already pointed out many similarities (as well as differences) between the two systems regarding their dynamics, since they both belong to the same generalized Lorenz family. If one is satisfied with the knowledge about their similarities, then there is no need to bother to look at the Chen system. But for those who have scientific curiosity trying to find out the subtle differences between the two topologically non-equivalent systems [3], perhaps one should respect their academic freedom rather than saying that their works are “unnecessary or incorrect” (said in the Abstract and the Lead Paragraph of [1]).

It is also worth mentioning that in a recent paper [7], similarly to [1], the same wrong arguments are used to conclude that the Lü system is a special case of the Lorenz system, but that two systems are defined on two very different parameter sets therefore they are not “equivalent”, as has been thoroughly discussed above.

As a concluding remark, it is not advisable to publish so many similar commentary articles [8-12] (which are Refs. 56-59 and 63 in [1]), in which all technical content is the same in nature. Actually, one can (in fact, should) publish only one commentary and then refer to it in future publications if necessary. This can relieve the burden of our academic journals as well.

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